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The Pricing Formula of Multi-stage Compound Option

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Abstract: As a kind of real options, the multi-stage compound option has become an important technique in valuation of venture capital. In this paper, the pricing formula of multi-stage compound option is derived by basing on the Black-Scholes model, and is expanded to the cases of continuous dividend and time-dependent parameter.

Keywords: multi-stage; compound option; real option; dividend; time-dependent parameter

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1 Introduction

The real option theory is often used in making investment decisions, which applies the financial option pricing theory to the investment decision analytical method. Multi-Stage Compound Option (MSCO) is a very important kind of real options. In the multi-stage venture capital, investors have the right to continue making the next stage investment in each decision point. Likewise, the holder of MSCO can obtain the next option with the strike price on each exercise date. Thus the study on the pricing method of MSCO is significant for investment decision making, especially for the value assessment of the asset-intensive long-term project.

Geske[1] proposed a pricing formula of the two-stage compound option by solving the partial-differential equation. In [2], the same formula is proved by the conditional probability method. This paper proposes the pricing formula of MSCO by basing on the Black-Scholes (B-S) model and utilizing the property of the conditional expectation and the matrix, which extends the conclusions of two stages to multi-stage. Further, the pricing formula of MSCO is discussed in cases of the dividend is paid stage by stage and the parameters are time depended.

2 Basic assumption

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a standard Brownian motion $(W_t)_{t\geq 0}$. We shall denote by $\mathcal{F}$ the natural filtration of W. Consider a B-S financial market in which we have two securities. For a bond $B=(B_t)_{t\geq 0}$, it is assumed that

$$B_t = \mathrm{e}^{rt}, \tag{1}$$

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where r is the deterministic interest rate. And for a risky security $S = (S_t)_{t\geq 0}$, we shall assume S is a random process with

$$S_t(\mu) = S_0 \exp\Big\{\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big\}, \tag{2}$$

where $\mu \in \mathbf{R}$ is an appreciation rate and $\sigma > 0$ is the volatility coefficient.

Assume that a multi-stage compound option O_0 has n exercise dates $0 < t_1 < t_2 < \cdots < t_n$, with the strike prices K_i, $i = 1, 2, \cdots, n$, respectively. As a kind of call option, real option allows the holder to buy O_1 at t_1 with strike price K_1. Likely O_i allows the holder to buy O_{i+1} at t_{i+1} with price K_{i+1} and so on. Certainly, if the value of O_{i+1} is smaller than K_{i+1}, the holder can choose not to exercise the following options with $i = 0, 1, \cdots, n-1$. If the holder obtains the last stage option O_{n-1} after all stage options are exercised, he can choose to buy risk asset with price S_{t_n} at t_n with the price K_n. Let C_i be the value of option O_i at time t_i which exercised at t_{i+1}, $i = 0, 1, \cdots, n-1$. It is clearly that the payment functions

$$f_i = (C_{i+1} - K_{i+1})^+, \quad i = 0, 1, \cdots, n, \tag{3}$$

where $C_n = S_{t_n}$.

3 Main results and proof

The following lemmas are used in this article without proof.

Lemma 1[3] Let $f_T(\omega)$ be a natural payment function in the European option at the expiration time $T > 0$ such that $Ef_T^{1+\varepsilon}(\omega) < \infty$ for some $\varepsilon > 0$. Then the price at time t is determined by the formula

$$C_t = E\big(\mathrm{e}^{-r(T-t)} f_T(S_T(r)) \mid \mathcal{F}_t\big). \tag{4}$$

Lemma 2[4] Let $g(x, y)$ be a bivariate Borel measurable function, and it makes the following formula meaningful. while X is $\mathcal{G}$ measurable random variable, and Y is independent from $\mathcal{G}$. Then

$$E\big(g(X, Y) \mid \mathcal{G}\big) = E\big(g(x, Y)\big) \mid_{x=X}. \tag{5}$$

Lemma 3 Let $(W_t)_{t\geq 0}$ be a standard Brownian motion in a complete probability space $(\Omega, \mathcal{F}, P)$. Set

$$\xi_i \triangleq \frac{W_i}{\sqrt{t_i}}, \quad i = 1, 2, \cdots, n,$$

then for any $j \leq n$, the union distribution of $(\xi_1, \xi_2, \cdots, \xi_j)$ is standard j-variate normal distribution with the covariance matrix as follows

$$D_j = \Big(\sqrt{\frac{t_{i\wedge k}}{t_{i\vee k}}}\Big)_{j\times j}. \tag{6}$$

The main results of this paper are as follows.

Theorem 1 For any $\omega \in \Omega$, $C_i(\omega)$ is an increasing function of $S_{t_i}(\omega)$, $i = 0, 1, \cdots, n-1$.

Proof Reverse Induction Approach.

When $i = n-1$, by (3), (4) and (5),

$$\begin{aligned} C_{n-1} &= E\big(\mathrm{e}^{-r(t_n-t_{n-1})} f_{t_n} \mid \mathcal{F}_{t_{n-1}}\big) \\ &= E\big(\mathrm{e}^{-r(t_n-t_{n-1})} (S_{t_n} - K_n)^+ \mid \mathcal{F}_{t_{n-1}}\big) \\ &= E\big(\mathrm{e}^{-r(t_n-t_{n-1})} (x\mathrm{e}^{\sigma W_{t_n-t_{n-1}}+(r-\frac{\sigma^2}{2})(t_n-t_{n-1})} - K_n)^+\big) \mid_{x=S_{t_{n-1}}}. \end{aligned} \tag{7}$$

Since

$$E\Big(\mathrm{e}^{-r(t_n-t_{n-1})}\big(x\mathrm{e}^{\sigma W_{t_n-t_{n-1}}+(r-\frac{\sigma^2}{2})(t_n-t_{n-1})} - K_n\big)^+\Big)$$

is an increasing function of x, we have for any ω, $C_{n-1}(\omega)$ is an increasing function of $S_{t_{i+1}}(\omega)$.

Now we suppose the conclusion hold for $i+1$, and from (3), (4) and (5), we also obtain

$$\begin{aligned} C_i &= E\big(\mathrm{e}^{-r(t_{i+1}-t_i)} (C_{i+1}(S_{t_{i+1}}) - K_{i+1})^+ \mid \mathcal{F}_{t_i}\big) \\ &= E\Big(\mathrm{e}^{-r(t_{i+1}-t_i)}\big(C_{i+1}\big(x\mathrm{e}^{\sigma W_{t_{i+1}-t_i}+(r-\frac{\sigma^2}{2})(t_{i+1}-t_i)}\big) - K_{i+1}\big)^+\Big) \mid_{x=S_{t_i}}. \end{aligned} \tag{8}$$

It can be seen that for any ω,

$$x\mathrm{e}^{\sigma W_{t_{i+1}-t_i}+(r-\frac{\sigma^2}{2})(t_{i+1}-t_i)}$$

increases with x, and from supposing, $C_{i+1}(\omega)$ is an increasing function of $S_{t_{i+1}(\omega)}$, thus we have

$$E\Big(\mathrm{e}^{-r(t_{i+1}-t_i)}\big(C_{i+1}\big(x\mathrm{e}^{\sigma W_{t_{i+1}-t_i}+(r-\frac{\sigma^2}{2})(t_{i+1}-t_i)}\big) - K_{i+1}\big)^+\Big).$$

increases with x, this proves the conclusion exists for i and the result follows.

Theorem 2 Let $D_n = (a_{ij})_{n\times n}$ be a positive definite symmetric matrix such that

$$a_{ij} = \sqrt{\frac{t_{i\wedge j}}{t_{i\vee j}}}.$$

Then for any $x_i \in \mathbf{R}$,

$$\begin{aligned} &(x_1 - \sigma\sqrt{t_1},\, x_2 - \sigma\sqrt{t_2},\, \cdots,\, x_n - \sigma\sqrt{t_n}) D_n^{-1} (x_1 - \sigma\sqrt{t_1},\, x_2 - \sigma\sqrt{t_2},\, \cdots,\, x_n - \sigma\sqrt{t_n})^\tau \\ = &(x_1, x_2, \cdots, x_n) D_n^{-1} (x_1, x_2, \cdots, x_n)^\tau - 2\sigma\sqrt{t_n} x_n + \sigma^2 t_n. \end{aligned} \tag{9}$$

Proof Let A_{ij} be algebraic cofactor of a_{ij}. We have

$$A_{ij} = A_{ji} \tag{10}$$

and

$$\sum_{i=1}^{n} a_{ik} A_{ij} = \begin{cases} |D_n|, & j = k, \\ 0, & j \neq k. \end{cases} \tag{11}$$

Setting $k = n$, and by (10), (11) we obtain

$$\sum_{i=1}^{n} \sqrt{t_i} A_{ji} = \begin{cases} \sqrt{t_n}|D_n|, & j = n, \\ 0, & j \neq n. \end{cases} \tag{12}$$

Since

$$D_n^{-1}=\frac{D_n^*}{|D_n|}=\frac{1}{|D_n|}(A_{ij})_{n*n}^{\tau},$$

it is obvious that

$$\begin{aligned}(\sqrt{t_1},\sqrt{t_2},\cdots,\sqrt{t_n})D_n^{-1}&=\frac{1}{|D_n|}\Big(\sum_{i=1}^n\sqrt{t_i}A_{1i},\sum_{i=1}^n\sqrt{t_i}A_{2i},\cdots,\sum_{i=1}^n\sqrt{t_i}A_{ni}\Big)z\\&=(0,\cdots,0,\sqrt{t_n}),\end{aligned}$$

$$(\sigma\sqrt{t_1},\sigma\sqrt{t_2},\cdots,\sigma\sqrt{t_n})D_n^{-1}(x_1,x_2,\cdots,x_n)^{\tau}=\sigma\sqrt{t_n}x_n,$$

$$(x_1,x_2,\cdots,x_n)D_n^{-1}(\sigma\sqrt{t_1},\sigma\sqrt{t_2},\cdots,\sigma\sqrt{t_n})^{\tau}=(\sigma\sqrt{t_n}x_n)^{\tau}=\sigma\sqrt{t_n}x_n,$$

$$(\sigma\sqrt{t_1},\sigma\sqrt{t_2},\cdots,\sigma\sqrt{t_n})D_n^{-1}(\sigma\sqrt{t_1},\sigma\sqrt{t_2},\cdots,\sigma\sqrt{t_n})^{\tau}=\sigma^2t_n,$$

then we get the result of (9).

Theorem 3 The price of the multi-stage compound option O_0 is determined by the formula

$$C_0=S_0N^n(a_1,\cdots,a_n,D_n)-\sum_{j=1}^nK_j\mathrm{e}^{-rt_j}N^j(b_1,\cdots,b_j,D_j),\tag{13}$$

where $N^j(b_1,\cdots,b_j,D_j)$ is the standard j-variate normal distribution function with covariance matrix

$$D_j=\Big(\sqrt{\frac{t_{i\wedge k}}{t_{i\vee k}}}\Big)_{j\times j},$$

and

$$a_i=\frac{\ln\frac{S_0}{S_i^*}+(r+\frac{\sigma^2}{2})t_i}{\sigma\sqrt{t_i}},\quad i\le n-1,\qquad a_n=\frac{\ln\frac{S_0}{K_n}+(r+\frac{\sigma^2}{2})t_n}{\sigma\sqrt{t_n}},\tag{14}$$

$$b_i=\frac{\ln\frac{S_0}{S_i^*}+(r-\frac{\sigma^2}{2})t_i}{\sigma\sqrt{t_i}},\quad i\le n-1,\qquad b_n=\frac{\ln\frac{S_0}{K_n}+(r-\frac{\sigma^2}{2})t_n}{\sigma\sqrt{t_n}}.\tag{15}$$

S_i^* is the unique solution of the equation

$$C_i(S_{t_i})=0.\tag{16}$$

Proof According to Lemma 1 and Lemma 2, we deduce

$$\begin{aligned}C_0&=E(\mathrm{e}^{-rt_1}f_{t_1}\mid\mathcal{F}_{t_0})\\&=E\big(\mathrm{e}^{-rt_1}(C_1-K_1)^+\big)\\&=E\big(\mathrm{e}^{-rt_1}(E(\mathrm{e}^{-r(t_2-t_1)}(C_2-K_2)^+\mid\mathcal{F}_{t_1})-K_1)I_{(C_1\ge K_1)}\big)\\&=E\big(\mathrm{e}^{-rt_2}E((C_2-K_2)^+I_{(C_1\ge K_1)}\mid\mathcal{F}_{t_1})-K_1\mathrm{e}^{-rt_1}I_{(C_1\ge K_1)}\big)\\&=E\big(\mathrm{e}^{-rt_2}(C_2-K_2)I_{(C_1\ge K_1,C_2\ge K_2)}\big)-K_1\mathrm{e}^{-rt_1}E(I_{(C_1\ge K_1)})\\&=E\big(\mathrm{e}^{-rt_2}C_2I_{(C_1\ge K_1,C_2\ge K_2)}\big)-K_2\mathrm{e}^{-rt_2}E\big(I_{(C_1\ge K_1,C_2\ge K_2)}\big)-K_1\mathrm{e}^{-rt_1}E(I_{(C_1\ge K_1)}).\end{aligned}\tag{17}$$

Since

$$C_2 = E\big(\mathrm{e}^{-r(t_3-t_2)}(C_3 - K_3)^+ \mid \mathcal{F}_{t_2}\big),$$

we have

$$C_0 = E\big(\mathrm{e}^{-rt_3} C_3 I_{(C_1\ge K_1, C_2\ge K_2, C_3\ge K_3)}\big) - \sum_{j=1}^{3} K_j \mathrm{e}^{-rt_j} E I_{(C_1\ge K_1,\cdots,C_j\ge K_j)}. \tag{18}$$

Similarly, for

$$C_i = E\big(\mathrm{e}^{-r(t_{i+1}-t_i)}(C_{i+1}(S_{t_{i+1}}) - K_{i+1})^+ \mid \mathcal{F}_{t_i}\big),$$

we get finally

$$\begin{aligned} C_0 \;=\; & E\big(\mathrm{e}^{-rt_n} S_{t_n} I_{(C_1\ge K_1,\cdots,C_{n-1}\ge K_{n-1},S_{t_n}\ge K_n)}\big) \\ & -K_n \mathrm{e}^{-rt_n} E I_{(C_1\ge K_1,\cdots,C_{n-1}\ge K_{n-1},S_{t_n}\ge K_n)} \\ & -\sum_{j=1}^{n-1} K_j \mathrm{e}^{-rt_j} E I_{(C_1\ge K_1,\cdots,C_j\ge K_j)}. \end{aligned} \tag{19}$$

Let S_i^* be the unique solution of (16). Apply Theorem 1, firstly, we obtain

$$I_{(C_i\ge K_i)} = I_{(S_{t_i}\ge S_i^*)} = I\Big(\xi_i \ge \frac{\ln\frac{S_i^*}{S_0} - (r-\frac{\sigma^2}{2})t_i}{\sigma\sqrt{t_i}}\Big), \tag{20}$$

$$d_i = \frac{\ln\frac{S_i^*}{S_0} - (r-\frac{\sigma^2}{2})t_i}{\sigma\sqrt{t_i}}, \quad i\le n-1, \qquad d_n = \frac{\ln\frac{K_n}{S_0} - (r-\frac{\sigma^2}{2})t_n}{\sigma\sqrt{t_n}}. \tag{21}$$

Then by (19), (20) and (21)

$$C_0 = E\big(\mathrm{e}^{-rt_n} S_{t_n} I_{(\xi_1\ge d_1,\cdots,\xi_n\ge d_n)}\big) - \sum_{j=1}^{n} K_j \mathrm{e}^{-rt_j} E I_{(\xi_1\ge d_1,\cdots,\xi_j\ge d_j)}. \tag{22}$$

From Lemma 3, we deduce that for any $j\le n$

$$\begin{aligned} E I_{(\xi_1\ge d_1,\cdots,\xi_j\ge d_j)} &= P(\xi_1\ge d_1,\cdots,\xi_j\ge d_j) = P(-\xi_1\le -d_1,\cdots,-\xi_j\le -d_j) \\ &= N^j(-d_1,\cdots,-d_j,D_j) = N^j(b_1,\cdots,b_j,D_j). \end{aligned} \tag{23}$$

$$\begin{aligned} & E\big(\mathrm{e}^{-rt_n} S_{t_n} I_{(\xi_1\ge d_1,\cdots,\xi_n\ge d_n)}\big) \\ = & \int_{d_1}^{\infty}\mathrm{d}x_1 \int_{d_2}^{\infty}\mathrm{d}x_2 \cdots \int_{d_n}^{\infty} S_0 \mathrm{e}^{\sigma\sqrt{t_n}x_n - \frac{\sigma^2}{2}t_n} \frac{1}{(2\pi)^{\frac{n}{2}}|D_n|^{\frac{1}{2}}} \mathrm{e}^{-\frac{1}{2}(x_1,x_2,\cdots,x_n)D_n^{-1}(x_1,x_2,\cdots,x_n)^\tau}\mathrm{d}x_n \\ = & S_0 \int_{d_1}^{\infty}\mathrm{d}x_1 \int_{d_2}^{\infty}\mathrm{d}x_2 \cdots \int_{d_n}^{\infty} \frac{1}{(2\pi)^{\frac{n}{2}}|D_n|^{\frac{1}{2}}} \mathrm{e}^{-\frac{1}{2}((x_1,x_2,\cdots,x_n)D_n^{-1}(x_1,x_2,\cdots,x_n)^\tau + \sigma^2 t_n - 2\sigma\sqrt{t_n})}\mathrm{d}x_n. \end{aligned}$$

Let $y_i = x_i - \sigma\sqrt{t_i}$, and apply Theorem 2, it follows that

$$\begin{aligned}
&E\big(\mathrm{e}^{-rt_n} S_{t_n} I_{(\xi_1\geq d_1,\cdots,\xi_n\geq d_n)}\big)\\
&= S_0\int_{d_1-\sigma\sqrt{t_1}}^{\infty}\mathrm{d}y_1\int_{d_2-\sigma\sqrt{t_2}}^{\infty}\mathrm{d}y_2\cdots\int_{d_n-\sigma\sqrt{t_n}}^{\infty}\frac{1}{(2\pi)^{\frac{n}{2}}|D_n|^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{2}((y_1,y_2,\cdots,y_n)D_n^{-1}(y_1,y_2,\cdots,y_n)^{\tau})}\mathrm{d}y_n\\
&= S_0N^n(\sigma\sqrt{t_1}-d_1,\cdots,\sigma\sqrt{t_n}-d_n,D_n)\\
&= S_0N^n(a_1,\cdots,a_n,D_n).
\end{aligned}\tag{24}$$

By (22), (23) and (24),

$$C_0 = S_0N^n(a_1,\cdots,a_n,D_n)-\sum_{j=1}^{n}K_j\mathrm{e}^{-rt_j}N^j(b_1,\cdots,b_j,D_j).$$

The proof of Theorem 3 is finished.

4 Further discussion

About the calculation

The price of MSCO can be calculated by the formula (13). The only unclear variable is S_i^*, it is the unique solution to (16). By (7) and (8), we can deduce the function C_{n-1} of $S_{t_{n-1}}$

$$\begin{aligned}
C_{n-1}(S_{t_{n-1}}) &= S_{t_{n-1}}\Phi\Big(\frac{\ln\frac{S_{t_{n-1}}}{K_n}+(r+\frac{\sigma^2}{2})(t_n-t_{n-1})}{\sigma\sqrt{t_n-t_{n-1}}}\Big)\\
&\quad -K_ne^{-r(t_n-t_{n-1})}\Phi\Big(\frac{\ln\frac{S_{t_{n-1}}}{K_n}+(r-\frac{\sigma^2}{2})(t_n-t_{n-1})}{\sigma\sqrt{t_n-t_{n-1}}}\Big)
\end{aligned}\tag{25}$$

and the recursion formula

$$\begin{aligned}
C_i(S_{t_i}) &= \int_{-\infty}^{+\infty}\mathrm{e}^{-r(t_{i+1}-t_i)}\big(C_{i+1}(x\mathrm{e}^{\sigma y+(r-\frac{\sigma^2}{2})(t_{i+1}-t_i)})-K_{i+1}\big)^+\\
&\quad \frac{1}{\sqrt{2\pi(t_{i+1}-t_i)}}\mathrm{e}^{-\frac{y^2}{2(t_{i+1}-t_i)}}\mathrm{d}y\ |_{x=S_{t_i}}\ .
\end{aligned}\tag{26}$$

Then the function can be obtained and S_i^* can be get from solving the equation (16). The precision in calculation depends on numerical computation.

About the type compounded of call and put options

The Multi-stage compound option is a kind of real option. So we give the pricing formula of the type compounded of call options in this paper. The pricing formula of other types compounded of call and put options can also be deduced by the same approach.

About the case of continuous dividend

Consider the case that continuous dividend is paid stage by stage and let q_i be the dividend rate between $[t_i,t_{i+1}]$. Thus Lemma 1 can be expanded as following[5]

$$C_t = E\big(\mathrm{e}^{-r(T-t)}f_T(S_T(r-q))\mid\mathcal{F}_t\big).\tag{27}$$

Therefore

$$C_i = E\big(\mathrm{e}^{-r(t_{i+1}-t_i)}(C_{i+1}(S_{t_{i+1}}(r-q_i)) - K_{i+1})^+ \mid \mathcal{F}_{t_i}\big), \tag{28}$$

Prove similarly to Theorem 3, except for

$$I_{(C_i \geq K_i)} = I_{(S_{t_i} \geq S_i^*)} = I\Big(\xi_i \geq \frac{\ln\frac{S_i^*}{S_0} - (r - q_i - \frac{\sigma^2}{2})t_i}{\sigma\sqrt{t_i}}\Big). \tag{29}$$

Change r in formula (14), (15) and S_{t_i} in equation (16) to $r - q_i$, then (13) becomes the pricing formula in case of paying continuous dividend.

About the case of time-dependent parameter

In the above, we assumed the parameters were constants. But in fact the interest rate, the dividend rate and the volatility coefficient often change. Consider that the parameters are functions $\mu(s)$, $\sigma(s)$, $r(s)$, $q_i(s)$ only dependent on time and then

$$B_t = \mathrm{e}^{\int_0^t r(s)\mathrm{d}s}, \tag{30}$$

$$S_t(\mu) = S_0 \exp\Big\{\int_0^t \Big(\mu(s) - \frac{\sigma^2(s)}{2}\Big)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W_s\Big\}. \tag{31}$$

Using the Girsanov theorem to construct equivalent martingale measure, the hedging strategy of European option is obtained[6]. We have

$$C_i = E\big(\mathrm{e}^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s}(C_{i+1}(S_{t_{i+1}}) - K_{i+1})^+ \mid \mathcal{F}_{t_i}\big). \tag{32}$$

Then using the same way we deduced the pricing formula as follows

$$C_0^* = S_0 N^n(a_1^*, \cdots, a_n^*, D_n^*) - \sum_{j=1}^n K_j \mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} N^j(b_1^*, \cdots, b_j^*, D_j^*), \tag{33}$$

where

$$a_i^* = \frac{\ln\frac{S_0}{S_i^*} + \int_0^{t_i}(r(s) - q_i(s) + \frac{\sigma^2(s)}{2})\mathrm{d}s}{\sqrt{\int_0^{t_i}\sigma^2(s)\mathrm{d}s}}, \quad i \leq n-1,$$

$$a_n^* = \frac{\ln\frac{S_0}{K_n} + \int_0^{t_n}(r(s) - q_n(s) + \frac{\sigma^2(s)}{2})\mathrm{d}s}{\sqrt{\int_0^{t_n}\sigma^2(s)\mathrm{d}s}}, \tag{34}$$

$$b_i^* = \frac{\ln\frac{S_0}{S_i^*} + \int_0^{t_i}(r(s) - q_i(s) - \frac{\sigma^2(s)}{2})\mathrm{d}s}{\sqrt{\int_0^{t_i}\sigma^2(s)\mathrm{d}s}}, \quad i \leq n-1,$$

$$b_n^* = \frac{\ln\frac{S_0}{K_n} + \int_0^{t_n}(r(s) - q_n(s) - \frac{\sigma^2(s)}{2})\mathrm{d}s}{\sqrt{\int_0^{t_n}\sigma^2(s)\mathrm{d}s}}, \tag{35}$$

$D_j^* = (a_{ik}^*)_{j\times j}$, and

$$a_{ik}* = \sqrt{\frac{\int_0^{t_i \wedge k}\sigma^2(s)\mathrm{d}s}{\int_0^{t_i \vee k}\sigma^2(s)\mathrm{d}s}}. \tag{36}$$

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多阶段复合期权的定价方法

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摘　要: 多阶段复合期权是实物期权的一种，它已经成为投资决策的重要方法。本文在B-S模型的基础上，推导出了多阶段复合期权的定价公式，并针对分阶段支付红利以及参数依赖于时间等情形对公式进行了推广。

关键词: 多阶段；复合期权；实物期权；红利；参数依赖时间